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# Steady ablation on the surface of a two-layer composite

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**Abstract**

Discovered is a quasi-steady ablation phenomenon on the surface of a two-layer composite which is formed by a layer of ablative material and another layer of non-ablative substrate. Theoretical exact solutions of quasi-steady ablation rate, the associated temperature distribution and end-of-ablation time of this two-layer composite are derived. A criterion for the occurrence of quasi-steady ablation is presented also. A one-dimensional transient numerical model is developed to perform a number of numerical experiments and hence to verify the correctness of the above theoretical solutions for the current quasi-steady ablation phenomenon. Based on the current results, a new method of measuring the ablation (or sublimation) heat is also proposed.


*Keywords:* Ablation; Sublimation; Steady state; Theoretical solutions

## 1. Introduction

The term "ablation" usually refers to the removal of mass from the surface of a body by thermochemical and mechanical processes. There are many applications involving ablation. Typical examples are thermal protection systems of space vehicles or missiles, melting or sublimation of a solid, medical surgeries and micro-machining using lasers, etc. Transient heat transfer in solids involving ablation is basically non-linear and general exact transient solutions to this kind of problems are not available. Even for the very simple idealized case of Landau [1], which is appropriate for the ablation of subliming materials, only a special case of large latent heat could be solved for the exact transient solutions. Since Landau, in 1950, presented his exact solutions [1], no other theoretical progress is achieved and it is, to date, the only available set of exact solutions for ablation in the literature.

In view of the difficulties of obtaining the exact solutions, some analytical approximation methods for solving this kind of problems were proposed by Goodman [2,3] and Zien [4,5]. Though these methods were shown to be very useful in a few simple cases, but in general, they could be very laborious and sometimes the numerical calculations are needed since no solution in closed analytical form could be obtained. There has been a good review of the related literature by Potts [6].

In the present work, the quasi-steady (or just referred as "steady" for brevity in the rest of this paper) ablation process on the surface of a two-layer composite is investigated theoretically. To the best of the author's knowledge, the exact steady solution for this problem does not seem to have been reported in the literature. The two-layer composite is consisting of a layer of ablative material combined with another layer of non-ablative

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**Nomenclature**

| | |
|---|---|
| $A$ | $A \equiv \alpha_1/\alpha_2$, ratio of thermal diffusivity |
| $c_{p1}$ | specific heat of ablative material, J/(kg °C) |
| $c_{p2}$ | specific heat of substrate, J/(kg °C) |
| $h_{ab}$ | ablation heat, J/kg |
| $H$ | $H \equiv \rho_1 h_{ab}\alpha_2/(q_{ex}L_2)$, dimensionless ablation heat |
| $k_1$ | thermal conductivity of ablative material, W/(m °C) |
| $k_2$ | thermal conductivity of substrate, W/(m °C) |
| $K$ | $K \equiv k_1/k_2$, ratio of thermal conductivity |
| $L_2$ | thickness of substrate, m |
| $q_{ex}$ | external heat flux imposed on the heated surface, W/m$^2$ |
| $q_{int}$ | heat flux at the interface between two materials, W/m$^2$ |
| $Q_{int}$ | $Q_{int} \equiv q_{int}/q_{ex}$, dimensionless interfacial heat flux |
| $s$ | residual thickness of ablative material, m |
| $s_0$ | initial thickness of ablative material, m |
| $S$ | $S \equiv s/L_2$, dimensionless residual thickness of ablative material |
| $S_0$ | $S_0 \equiv s_0/L_2$, dimensionless initial thickness of ablative material |
| $S_c$ | dimensionless residual thickness of ablative material when $T = T_{s2}$ |
| $t$ | time, s |
| $t_e$ | end-of-ablation time, s |
| $T$ | $T \equiv \alpha_2 t/L_2^2$, dimensionless time |
| $T_e$ | $T_e \equiv \alpha_2 t_e/L_2^2$, dimensionless end-of-ablation time |
| $T_{s2}$ | dimensionless time at the start of quasi-steady ablation |
| $u_1$ | temperature of ablative material, °C |
| $u_2$ | temperature of substrate, °C |
| $u_{ab}$ | ablation temperature, °C |
| $u_b$ | backside temperature of substrate, °C |
| $u_i$ | initial temperature, °C |
| $u_{int}$ | temperature at the interface, °C |
| $u_s$ | surface temperature at $y = 0$, °C |
| $U_1$ | $U_1 \equiv (u_1 - u_{ab})/[q_{ex}L_2/k_1]$, dimensionless temperature of ablative material |
| $U_2$ | $U_2 \equiv (u_2 - u_{ab})/[q_{ex}L_2/k_1]$, dimensionless temperature of substrate |
| $U_b$ | $U_b \equiv U_2\|_{X=1}$, dimensionless backside temperature of substrate |
| $U_i$ | $U_i \equiv (u_i - u_{ab})/[q_{ex}L_2/k_1]$, dimensionless initial temperature |
| $U_{int}$ | $U_{int} \equiv (u_{int} - u_{ab})/[q_{ex}L_2/k_1]$, dimensionless interfacial temperature |
| $U_s$ | $U_s \equiv (u_s - u_{ab})/[q_{ex}L_2/k_1]$, dimensionless temperature on the heated surface |
| $v$ | ablation rate, m/s |
| $v_e$ | ablation rate at the end-of-ablation time, m/s |
| $v_{s1}$ | steady ablation rate proposed by Landau, m/s |
| $v_{s2}$ | steady ablation rate proposed in the present work, m/s |
| $v_{sat}$ | saturated ablation rate, m/s |
| $V$ | $V \equiv vL_2/\alpha_2$, dimensionless ablation rate |
| $V_e$ | $V_e \equiv v_e L_2/\alpha_2$, dimensionless ablation rate at the end-of-ablation time |
| $V_{s1}$ | $V_{s1} \equiv v_{s1}L_2/\alpha_2$, dimensionless steady ablation rate proposed by Landau |
| $V_{s2}$ | $V_{s2} \equiv v_{s2}L_2/\alpha_2$, dimensionless steady ablation rate proposed in the present work |
| $V_{sat}$ | $V_{sat} \equiv v_{sat}L_2/\alpha_2$, dimensionless saturated ablation rate |
| $x$ | axial distance from the interface, m |
| $X$ | $X \equiv x/L_2$, dimensionless axial distance from the interface |
| $y$ | axial distance from the heated surface, m |
| $Y$ | $Y \equiv y/L_2$, dimensionless axial distance from the heated surface |

*Greek symbols*

| | |
|---|---|
| $\alpha_1$ | thermal diffusivity of ablative material, m$^2$/s |
| $\alpha_2$ | thermal diffusivity of substrate, m$^2$/s |
| $\rho_1$ | density of ablative material, kg/m$^3$ |
| $\rho_2$ | density of substrate, kg/m$^3$ |

*Subscripts*

| | |
|---|---|
| 0 | initial |
| 1 | ablative material |
| 2 | substrate |
| e | end-of-ablation |
| ex | external |
| i | initial |
| int | interface |
| s | surface |
| s1 | steady state proposed by Landau |
| s2 | steady state proposed in the present work |
| sat | saturated |

substrate which is usually a backup structure. Materials in both layers are assumed to be homogeneous and isotropic and in good thermal contact with each other. This two-layer composite is assumed initially at a uniformly distributed temperature and then suddenly exposed to a constant external heat flux on the outer surface of the ablative material so that a phase change occurs and the new phase is removed immediately on formation.

The external heat flux could be provided by aerodynamic heating or by radiation. While ablation continues on the heated surface of ablative material. It is assumed that the surface temperature is kept at a constant ablation temperature which is lower than the melting temperature of the substrate. The rear face of the non-ablative substrate is assumed to be adiabatic. All thermal properties involved in the present work are assumed to be constant. Thus, the problem currently investigated is basically the Landau's idealized ablation model [1] combined with a substrate of finite thickness.

The main purpose of this paper is to present the exact solutions of quasi-steady ablation for this problem. In Section 2, The governing equations are introduced first and then transformed to dimensionless forms for brevity. Theoretical analyses are performed in Section 3 to obtain the exact solutions of steady ablation rate and the associated temperature distribution in the two-layer composite. Also derived are the solution of end-of-ablation time and a criterion of the occurrence of steady ablation. In Section 4, a number of numerical experiments are conducted to verify the derived exact solutions and finally the conclusions are presented in Section 5. The exact solutions of steady ablation obtained in this research are useful as benchmark data for numerical models associated with similar non-linear problems.

## 2. Mathematical formulation

### 2.1. Governing equations

Consider a two-layer composite formed by a layer of ablative material and another layer of non-ablative substrate. The ablative material is considered to be acting as an insulation for the substrate whereas the latter acting as a heat sink for the former. As shown in Fig. 1, the origin of $y$ coordinate is fixed on the heated surface of the ablative material, thus, the materials are considered to move toward the heated surface in a velocity of ablation rate. For convenience, the origin of $x$ coordinate is set to locate at the interface between the ablative material and its substrate, i.e., at $y = s$, thus

$$x \equiv y - s \tag{1}$$

Therefore, the governing equation for the temperature of ablative material is derived to be

$$\frac{\partial^2 u_1}{\partial y^2} + (v/\alpha_1)\frac{\partial u_1}{\partial y} = \frac{1}{\alpha_1}\frac{\partial u_1}{\partial t} \tag{2}$$

Note that the second term of Eq. (2) arises from the moving of material toward the heated surface. Similarly, for the temperature of substrate, we have

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{1}{\alpha_2}\frac{\partial u_2}{\partial t} \tag{3}$$

This is a conventional equation of heat conduction.

The boundary conditions for the above two equations are

(A) at the heated surface ($y = 0$),

$$q_{ex} = -k_1 \frac{\partial u_1}{\partial y}\Big|_{y=0} \quad \text{if } u_s < u_{ab} \tag{4}$$

or

$$u_s = u_{ab} \tag{5}$$

(B) at the interface $y = s$, i.e., at $x = 0$,

$$u_1 = u_2 = u_{int}, \tag{6}$$

$$-k_1 \frac{\partial u_1}{\partial y}\Big|_{y=s} = -k_2 \frac{\partial u_2}{\partial x}\Big|_{x=0} = q_{int} \tag{7}$$

(C) the rear face of substrate is assumed to be adiabatic

$$\frac{\partial u_2}{\partial x}\Big|_{x=L_2} = 0 \tag{8}$$

The initial temperature of both ablative material and its substrate is assumed to be uniformly distributed, thus, at time $t = 0$, we have

$$u_1 = u_2 = u_i \tag{9}$$

For the thickness of ablative material, the following governing equation is given according to energy conservation:

$$q_{ex} = -k_1 \frac{\partial u_1}{\partial y}\Big|_{y=0} - \rho_1 h_{ab}\frac{ds}{dt} \quad \text{if } u_s = u_{ab} \tag{10}$$

If the temperature of the heated surface is lower than ablation temperature, there would be no ablation, i.e.

$$\frac{ds}{dt} = 0 \quad \text{if } u_s < u_{ab} \tag{11}$$

The initial condition for the thickness of ablative material is given as

$$s|_{t=0} = s_0 \tag{12}$$

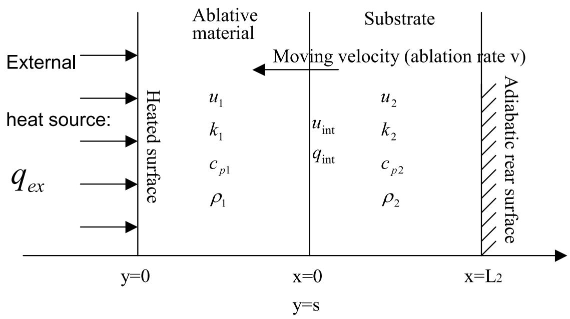


Fig. 1. Schematic of the ablation/sublimation problem.

The ablation rate (i.e., the moving velocity of material) $v$ in Eq. (2) is defined as

$$v \equiv -\frac{ds}{dt} \tag{13}$$

Thus, the ablation rate is considered to be always non-negative.

Eqs. (6) and (7) represent the continuity of temperature and heat flux at the interface between the ablative material and its substrate.

For Landau's idealized case, i.e., the thickness of the ablative material is assumed to be infinite, if ablation continues long enough, there are steady solutions for ablation rate and temperature distribution. Hence, by substituting $\frac{\partial u_1}{\partial t} = 0$ into Eq. (2), the steady temperature distribution is solved to be

$$\frac{(u_1 - u_i)}{(u_{ab} - u_i)} = \exp(-v_{s1} y/\alpha_1) \tag{14}$$

where $v_{s1}$ is the Landau's steady ablation rate which can be solved as

$$v_{s1} = \frac{q_{ex}}{\rho_1 [h_{ab} + c_{p1}(u_{ab} - u_i)]} \tag{15}$$

Note that the above results of Eqs. (14) and (15) were actually presented earlier by Landau [1] but in different notations.

### 2.2. Dimensionless equations

For brevity, Eqs. (1)–(12) can be expressed as the following dimensionless equations:

$$X \equiv Y - S \tag{1a}$$

Temperature in ablative material

$$A\frac{\partial^2 U_1}{\partial Y^2} + V\frac{\partial U_1}{\partial Y} = \frac{\partial U_1}{\partial T} \tag{2a}$$

Temperature in substrate

$$\frac{\partial^2 U_2}{\partial X^2} = \frac{\partial U_2}{\partial T} \tag{3a}$$

Boundary conditions are

(A) at the heated surface $Y = 0$,

$$\frac{\partial U_1}{\partial Y}\Big|_{Y=0} = -1 \quad \text{if } U_s < 0 \tag{4a}$$

or

$$U_s = 0 \tag{5a}$$

(B) at $Y = S$, i.e., at $X = 0$,

$$U_1 = U_2 = U_{int} \tag{6a}$$

$$-K\frac{\partial U_1}{\partial Y}\Big|_{Y=S} = -\frac{\partial U_2}{\partial X}\Big|_{X=0} = Q_{int}K \tag{7a}$$

(C) at the rear face of substrate, $X = 1$,

$$\frac{\partial U_2}{\partial X}\Big|_{X=1} = 0 \text{ (adiabatic)} \tag{8a}$$

Initial condition (when time $T = 0$) is given as

$$U_1 = U_2 = U_i \tag{9a}$$

The governing equation of ablative material's thickness $S$ is

$$\frac{\partial U_1}{\partial Y}\Big|_{Y=0} + H\frac{dS}{dT} + 1 = 0 \quad \text{if } U_s = 0 \tag{10a}$$

$$\frac{dS}{dT} = 0 \quad \text{if } U_s < 0 \tag{11a}$$

The initial condition of the ablative material's thickness is

$$S|_{T=0} = S_0 \tag{12a}$$

Observe that, from Eqs. (2a)–(12a), the three independent variables of this system are $X$, $Y$, and $T$, and the three constant parameters, namely, $A$, $K$, and $H$, would govern the behavior of the system, whereas $U_i$ and $S_0$ are two parameters associated with the initial conditions of the system.

Similarly, the dimensionless forms of Eqs. (13)–(15) are obtained respectively as following:

$$V \equiv -\frac{dS}{dT} \tag{13a}$$

$$\frac{U_1}{U_i} = 1 - \exp(-V_{s1}Y/A) \tag{14a}$$

$$V_{s1} = \frac{A}{HA - U_i} \tag{15a}$$

## 3. Theoretical analyses

Both results of Eqs. (14a) and (15a) for steady ablation are based upon Landau's idealized assumption that the thickness of ablative material is infinite. However, for more practical cases, the ablative material is usually used as a thermal insulation for its backup structure and has a finite thickness. Therefore, the effect of the substrate cannot be neglected all the time. In this situation, there are possibilities for the occurrence of another type of steady ablation which is different from that of Landau. This section is aimed at the derivations of theoretical solutions for this unique phenomenon and the physical details of its occurrence.

### 3.1. Temperature of ablative material

It is assumed that $U_1$ in Eqs. (2a)–(12a), under some circumstances, has steady solution for $Y$ coordinate, i.e.

$$\frac{\partial U_1}{\partial T} = 0 \tag{16}$$

Since $U_1$ in the above equation is assumed to be independent of time $T$, it is concluded, from Eq. (10a), that the ablation rate is also steady and is represented by $V_{s2}$. Thus, Eq. (2a) becomes:

$$\frac{d^2U_1}{dY^2} = -(V_{s2}/A)\frac{dU_1}{dY} \tag{17}$$

By using Eq. (10a) and the boundary condition of Eq. (5a), the temperature distribution in the ablative material can be solved easily from Eq. (17):

$$U_1 = \frac{A}{V_{s2}}(HV_{s2} - 1)[1 - \exp(-YV_{s2}/A)] \tag{18}$$

By knowing that $U_1 = U_{int}$ at position $Y = S$, the temperature at the interface is obtained from the above equation:

$$U_{int} = \frac{A}{V_{s2}}(HV_{s2} - 1)[1 - \exp(-SV_{s2}/A)] \tag{19}$$

If the dimensionless residual thickness $S$ is infinite, thus by setting $V_{s2} = V_{s1}$ and $U_{int} = U_i$, Eqs. (18) and (19) would then be reduced to Eqs. (14a) and (15a) respectively. Hence, the present results are fully consistent with those by Landau [1].

### 3.2. Quasi-steady ablation rate

Differentiating Eq. (19) with respect to time $T$ results in the following equation for temperature increasing rate at the interface:

$$dU_{int}/dT = V_{s2}(1 - HV_{s2})\exp(-SV_{s2}/A) \tag{20}$$

The heat flux at the interface can be obtained from Eq. (18)

$$Q_{int} = -dU_1/dY|_{Y=S} = (1 - HV_{s2})\exp(-SV_{s2}/A) \tag{21}$$

Note that the residual thickness $S$ in the above equation is a function of time, hence, $Q_{int}$ is also a function of time.

According to Ref. [7, p. 113], if the input heat flux to the substrate at $X$=0 is kept steady as $Q_{int} = 1$, the temperature in the substrate could be solved from Eqs. (3), (7), (8) and (9)

$$\bar{U}(T,X) = T + \frac{X^2}{2} - X + \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\exp(-n^2\pi^2 T)\cos(n\pi X)}{n^2} \tag{22}$$

where $\bar{U} \equiv \frac{(U_2 - U_i)}{K}$ represents the temperature response of substrate to a single unit step input. Observe that the exponential terms will decay out as dimensionless time $T$ becomes large. In fact, for $T$ larger than 1, the exponential terms in the above are negligible.

In our problem, $Q_{int}$ is obviously not steady. However, the solution in Eq. (22) can be used to construct the exact solution of our problem. According to the above equation and the Duhamel's theorem ([7, p. 159]), the exact solution of $U_2$ can be obtained as the following for a time-varying $Q_{int}$:

$$\frac{(U_2 - U_i)}{K} = \int_{\tau=0}^{T} Q_{int}(\tau)\frac{\partial\bar{U}(T-\tau,X)}{\partial T}d\tau \tag{23}$$

Assuming that the ablation rate remains unsteady until time $T_{s2}$, thus, for time $T \geqslant T_{s2}$, the integration in the above equation can be separated into two parts, i.e.,

$$\frac{(U_2 - U_i)}{K} = \int_{\tau=0}^{T_{s2}} Q_{int}(\tau)\frac{\partial\bar{U}(T-\tau,X)}{\partial T}d\tau + \int_{\tau=T_{s2}}^{T} Q_{int}(\tau)\frac{\partial\bar{U}(T-\tau,X)}{\partial T}d\tau \tag{24}$$

The first part ($\tau = 0 \sim T_{s2}$) cannot be obtained explicitly since there is no available transient solution for $Q_{int}$. On the other hand, the second part ($\tau = T_{s2} \sim T$) of integral can be obtained by using the known solution of $Q_{int}$ in Eq. (21) for steady ablation. Hence, by substituting Eqs. (21) and (22) into the above equation, and assuming that $A/V_{s2}^2 \gg 1$ and $T > T_{s2} + 1$, we have

$$\frac{(U_2(T,X) - U_i)}{KQ_{int}(T)} = \frac{1}{Q_{int}(T)}\int_{\tau=0}^{T_{s2}} Q_{int}(\tau)d\tau + \frac{A}{V_{s2}^2}\left[1 - \exp\left(-\frac{(T-T_{s2})V_{s2}^2}{A}\right)\right] + \left(\frac{X^2}{2} - X + \frac{1}{3}\right) \tag{25}$$

Note that the last assumption ($T > T_{s2} + 1$) is not necessary but just made for the simplicity of the derivation, this will be explained later. The details for the derivation of the above equation are not shown here but are attached in Appendix A. Differentiating the above equation with respect to time $T$ and using the assumption $A/V_{s2}^2 \gg 1$ again, it is obtained that

$$\partial U_2/\partial T = KQ_{int} \tag{26}$$

It should be noted that the substrate's temperature increasing rate in Eq. (26) is independent of position. This means that, under some situations, the variation of $Q_{int}$ with respect to time is so slow, compared with the heat transfer speed in the substrate, that $Q_{int}$ can be regarded as quasi-steady and the whole substrate would therefore has a uniform temperature increasing rate which is proportional to $Q_{int}$.

Substituting $X = 0$ into the above equation, we have $dU_{int}/dT = KQ_{int}$. Combining this expression with Eqs. (20) and (21) results in the following equation:

$$(V_{s2} - K)(1 - HV_{s2})\exp(-SV_{s2}/A) = 0 \tag{27}$$

There are two possible solutions for the above equation. The first one ($V_{s2} = 1/H$) is called the "saturated ablation rate" which is corresponding to the special case that initial temperature just equals the ablation temperature, i.e., $U_1 = U_2 = U_i = 0$. This special solution can also be obtained easily by solving the energy equation, Eq. (10a). The second solution

$$V_{s2} = K \tag{28}$$

is the new discovered one and is different from that of Landau (Eq. (15a)).

The above equation can be rewritten by using original parameters

$$v_{s2} = k_1/(\rho_2 c_{p2} L_2) \tag{28a}$$

It is observed from this equation that the steady ablation rate is a function of physical properties and the thickness of substrate. Hence the ablation rate is constant and not a function of time, this is consistent with the assumption of a steady state (Eq. (16)). In view of the above equation, it is also known that the steady ablation rate is independent of the external heat flux, ablation heat, ablation temperature, ablative material's initial thickness, and the residual thickness. Hence, once the steady ablation is achieved, it will be sustained to the end-of-ablation, i.e., the residual thickness of ablative material *s* will decrease linearly with time until the complete recession of ablative material, and the temperature at the interface will increase with time and approach the ablation temperature. Though Eqs. (28) and (28a) are derived on the assumption of $T > T_{s2} + 1$, they are actually applicable for $T \geqslant T_{s2}$ because it is impossible to have another steady solution for the time period $(T_{s2} + 1) \geqslant T \geqslant T_{s2}$ or else the solution will be discontinued and the assumption of steady ablation for $T \geqslant T_{s2}$ cannot be satisfied.

By introducing Eq. (28) into Eqs. (18)–(21), explicitly obtained are the following equations of temperature distribution in ablative material, interfacial temperature and its increasing rate, and interfacial heat flux for steady ablation

$$U_1 = \frac{A}{K}(HK - 1)[1 - \exp(-YK/A)] \tag{18a}$$

$$U_{\rm int} = \frac{A}{K}(HK - 1)[1 - \exp(-SK/A)] \tag{19a}$$

$$\mathrm{d}U_{\rm int}/\mathrm{d}T = K(1 - HK)\exp(-SK/A) \tag{20a}$$

$$Q_{\rm int} = (1 - HK)\exp(-SK/A) \tag{21a}$$

As shown in Eq. (18a), under the condition of steady ablation, the temperature of ablative material $U_1$ is a function of $Y$ only and is independent of time. This is consistent again with the assumption of a steady state (Eq. (16)).

### 3.3. Temperature of substrate

By substituting Eq. (21a) into Eq. (26), the temperature increasing rate of substrate for steady ablation can be expressed explicitly as

$$\partial U_2/\partial T = K(1 - HK)\exp(-SK/A) \tag{26a}$$

This can be rewritten by using original parameters

$$\partial u_2/\partial t = \left(q_{\rm ex} - \frac{k_1\rho_1 h_{\rm ab}}{\rho_2 c_{p2} L_2}\right)\frac{1}{\rho_2 c_{p2} L_2}\exp\left(-\frac{\rho_1 c_{p1} s}{\rho_2 c_{p2} L_2}\right) \tag{29}$$

As shown in Eq. (29), under the condition of steady ablation, the temperature of substrate is not steady however and the temperature increasing rate in substrate is independent of position and will become larger gradually to an upper limit while the residual thickness approaches zero. Note that this upper limit is independent of ablation temperature.

By introducing $X = 0$ into Eq. (25), and knowing that $U_2|_{X=0} = U_{\rm int}$, the following equation is obtained:

$$\frac{(U_{\rm int}(T) - U_i)}{KQ_{\rm int}(T)} = \frac{1}{Q_{\rm int}(T)}\int_{\tau=0}^{T_{s2}} Q_{\rm int}(\tau)\,\mathrm{d}\tau + \frac{A}{V_{s2}^2}\left[1 - \exp(-\frac{(T - T_{s2})V_{s2}^2}{A})\right] + \frac{1}{3} \tag{30}$$

Combining the above equation with Eq. (25), and eliminating $\frac{1}{Q_{\rm int}(T)}\int_{\tau=0}^{T_{s2}} Q_{\rm int}(\tau)\,\mathrm{d}\tau + \frac{A}{V_{s2}^2}[1 - \exp(-\frac{(T-T_{s2})V_{s2}^2}{A})] + \frac{1}{3}$, gives the temperature distribution of substrate for steady ablation

$$U_2 = U_{\rm int} + KQ_{\rm int}\left(\frac{X^2}{2} - X\right) \tag{31}$$

where $U_{\rm int}$ and $Q_{\rm int}$ can be obtained respectively from Eqs. (19a) and (21a). Note that this temperature distribution is a parabolic function of $X$.

Substituting $X$=1 into Eq. (31) gives the backside temperature of substrate

$$U_{\rm b} = U_{\rm int} - \frac{1}{2}KQ_{\rm int} \tag{32}$$

Thus, the backside temperature of substrate at the end-of-ablation time can be obtained

$$U_{\rm b}|_{T=T_{\rm e}} = -\frac{1}{2}K(1 - HK) \tag{33}$$

The above expression can be rewritten by using original parameters

$$u_{\rm b}|_{t=t_{\rm e}} - u_{\rm ab} = \frac{1}{2}\left(\frac{k_1\rho_1 h_{\rm ab}}{k_2\rho_2 c_{p2}} - \frac{q_{\rm ex}L_2}{k_2}\right) \tag{34}$$

In view of Eq. (29) or (34), a new method is proposed here for measuring the ablation (or sublimation) heat by means of measuring the backside temperature response if other parameters are known in advance.

### 3.4. Residual thickness and end-of-ablation time

For energy conservation of the whole system, we have

$$q_{ex}t = \rho_1(s_0 - s)[h_{ab} + c_{p1}(u_{ab} - u_i)] + \rho_1 c_{p1}\int_0^s (u_1 - u_i)\,dy + \rho_2 c_{p2}\int_0^{L_2}(u_2 - u_i)\,dx \tag{35}$$

After transforming, the following dimensionless equation is obtained from the above:

$$T = (S_0 - S)\left(H - \frac{U_i}{A}\right) + \frac{1}{A}\int_0^S (U_1 - U_i)\,dY + \frac{1}{K}\int_0^1 (U_2 - U_i)\,dX \tag{36}$$

Note that the first term on the right-hand side of the above equation represents the latent and sensible heat of ablative material that has been removed away by ablation. The second and third terms on the right-hand side represent respectively the sensible heat stored in the residual ablative material and substrate. If the steady ablation is achieved, thus, for any time $T \geqslant T_{s2}$, the temperature $U_1$ and $U_2$ in Eq. (36) can be obtained respectively from Eqs. (18a) and (31). Hence, Eq. (36) can be integrated and the following relationship between time $T$ and the residual thickness $S$ is obtained for steady ablation

$$T = S_0\left(H - \frac{U_i}{A}\right) - \frac{(S + U_i)}{K} - \frac{(1 - HK)}{3}\exp(-KS/A) \tag{37}$$

Furthermore, by substituting $S = 0$ into the above, we obtain the end-of-ablation time for steady ablation:

$$T_e = S_0 H - \frac{S_0 U_i}{A} - \frac{U_i}{K} - \frac{1}{3}(1 - HK) \tag{38}$$

From this expression, it can be concluded that, for steady ablation, the end-of-ablation time $T_e$ is a linear function of $S_0$, $U_i$, $H$, and $1/A$. By knowing that $HK<1$ (This will be proved later in Section 3.5)for steady ablation and if $T_e$ is much larger than 1, it is obvious that the last term in the above equation is negligible and $T_e$ is almost linearly dependent on $1/K$.

### 3.5. The criteria of the occurrence of quasi-steady ablation

In Section 3.2, it has been mentioned that, under some situations, the variation of $Q_{int}$ with respect to time is so slow, compared with the heat transfer speed in the substrate, that $Q_{int}$ can be regarded as quasi steady. In other words, the relative variation of $Q_{int}$ is very small, say,

$$(\Delta Q_{int})/Q_{int} \ll 1 \tag{39}$$

during a time period $\Delta T$ whereas $\Delta T$ must be large enough for substrate to achieve a steady temperature increasing rate. According to Eq. (22), $\Delta T$ may be larger than 1 for exponential terms to decay out, i.e.,

$$\Delta T \geqslant 1 \tag{40}$$

Dividing Eq. (39) by Eq. (40) and letting $\Delta T \to dT$ yield

$$\frac{1}{Q_{int}}\frac{dQ_{int}}{dT} \ll 1 \tag{41}$$

Substituting Eq. (21a) into the above, and knowing that $(-\frac{dS}{dT}) = V_{s2} = K$, we have the criterion

$$\frac{A}{K^2} \gg 1 \tag{42}$$

Note that, in the derivation of Eqs. (25) and (26), an assumption of $A/V_{s2}^2 \gg 1$ has been made. Hence, the criterion derived here is fully consistent with that assumption. Eq. (42) is believed to be one of the necessary criteria which must be satisfied for the occurrence of steady ablation. Hence, if the criterion of Eq. (42) is satisfied, it is possible but not always to achieve a steady ablation. On the contrary, if Eq. (42) is not satisfied, there would be no possibility of the occurrence of steady ablation.

The above equation can be rewritten by using original parameters

$$\frac{k_2\rho_2 c_{p2}}{k_1\rho_1 c_{p1}} \gg 1 \tag{43}$$

In view of the above, it can be deduced that the steady ablation of the present study can not be achieved if the properties of substrate are close to those of ablative material ($\frac{k_2\rho_2 c_{p2}}{k_1\rho_1 c_{p1}} \cong 1$). For the same reason, if substrate does not exist ($k_2\rho_2 c_{p2} = 0$), the current steady ablation phenomenon would not occur either.

It has been known from Eq. (10a) or (15a) that, when the initial temperature is equal to ablation temperature, the ablation rate has its maximum as the saturated ablation rate ($V_{sat} = \frac{1}{H}$). This is the uppermost physical limitation for all possible cases, hence, $V_{s2} < V_{sat}$. After combining with Eq. (28), we have

$$HK < 1 \tag{44}$$

Moreover, by observing the results of Eqs. (18a)–(21a), it is also found that the value of $HK$ must be smaller than 1, or else these equations would lose their physical meanings. This is consistent with Eq. (44). It should be noted that Eq. (44) is not a necessary criterion for the occurrence of steady ablation but just a derived result while steady ablation is achieved.

It seems that the necessary criteria for steady ablation to occur are very complicated. Except the one ($A/K^2 \gg 1$) that is obtained in the above, there should be some other necessary criteria which must be all satisfied prior to the occurrence of steady ablation. More works should be done on this subject in the future.

## 4. Numerical experiments

Instead of conducting real experiments, a one-dimensional transient numerical model for solving Eqs. (2)–(12) is implemented in order to perform some numerical experiments and hence to verify the correctness of the theoretical solutions in the above section. The method of deforming grids is used for the ablative material in this finite-difference numerical model. Hence, during numerical calculations, the sizes and positions of nodes in the ablative material may vary with time due to the recession of ablative material. On the heated surface, the ablation rate is calculated explicitly whereas the temperature field is solved by fully implicit method. Landau's [1] exact solutions for steady ablation of an semi-infinite solid have been used to compare with those obtained from this numerical model. Highly consistent results, not shown here, are obtained and hence the numerical model is proved to be reliable and valid.

Shown in Figs. 2–6 are typical results of ablation rate by numerical experiments. From these figures, it is observed that the dimensionless ablation rate would approach a steady value of $K$ before the end-of-ablation. The variations of $A$, $H$, $U_i$, and $S_0$ are observed to have no influence on this steady ablation rate though these four parameters do have their effects on transient ablation. This is fully consistent with the theoretical prediction of Eq. (28).

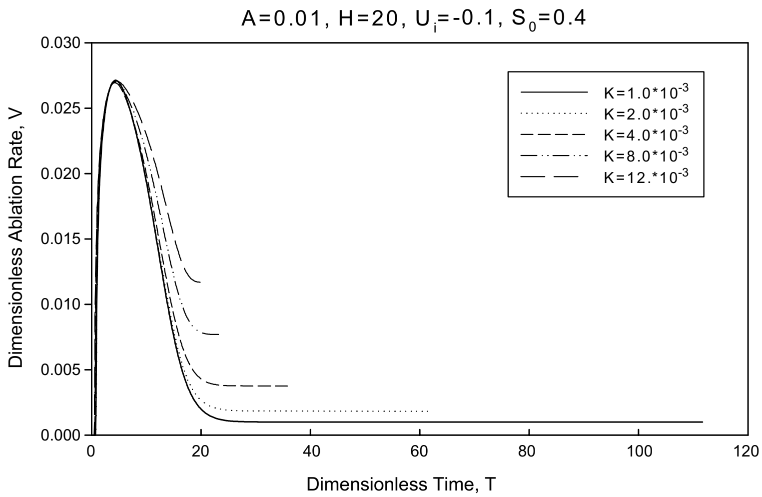


Fig. 2. Effects of thermal-conductivity ratio on the history of ablation rate, resulting from numerical experiments.

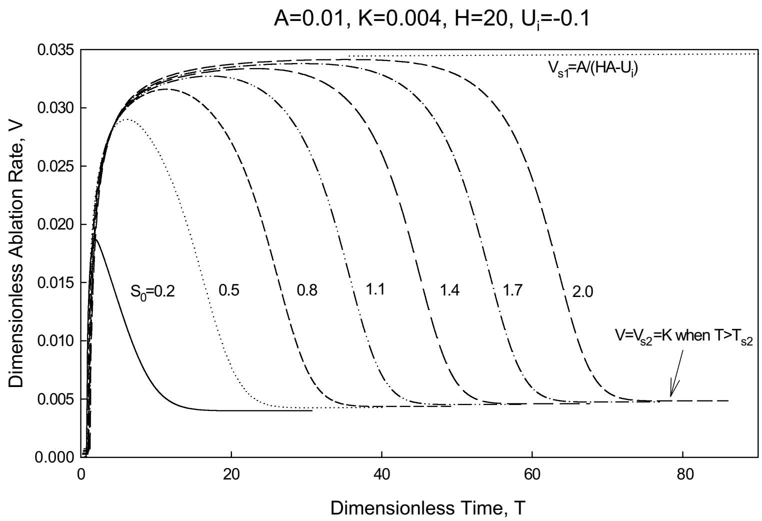


Fig. 3. Effects of initial thickness on the history of ablation rate, resulting from numerical experiments.

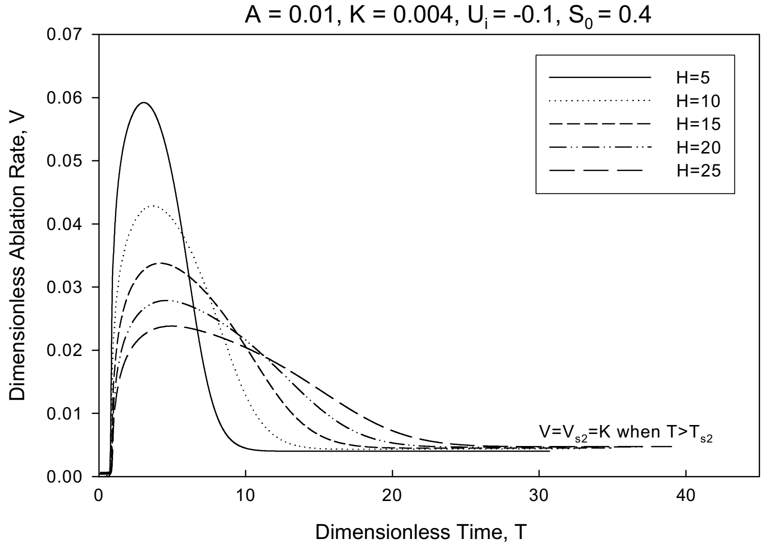


Fig. 4. Effects of dimensionless ablation heat on the history of ablation rate, resulting from numerical experiments.

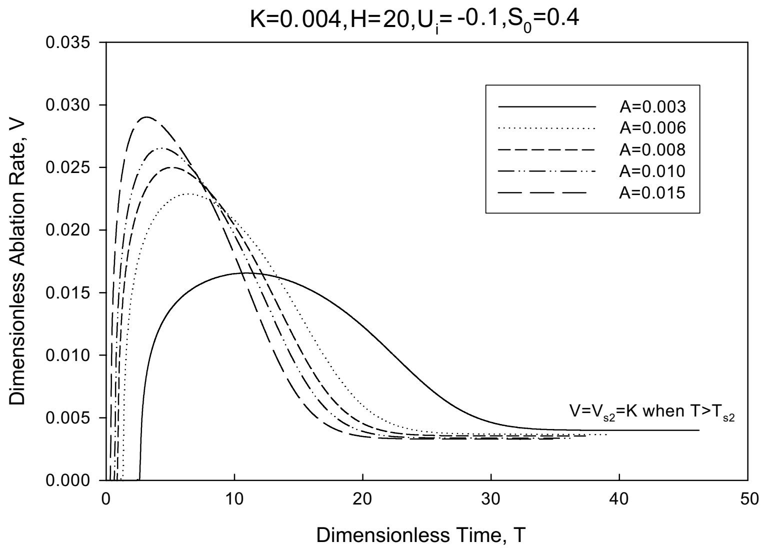


Fig. 5. Effects of thermal-diffusivity ratio on the history of ablation rate, resulting from numerical experiments.

It is also observed that, in the beginning, there is a short period of time of zero ablation rate since the surface temperature has not yet been raised to the ablation temperature. This time period of zero ablation rate is usually called the "pre-ablation period". After this pre-ablation period, the ablation rate increases with time and reaches a maximum at a later time. From the "experimental" results shown in Fig. 3, it is observed that this maximum ablation rate increases with the increasing of initial thickness, and finally approaches a limit corresponding to Landau's steady solution $V_{s1}$ (this is the horizontal dotted line in this figure). Beyond that, more increasing of initial thickness would have no further influence on the maximum ablation rate and the steady ablation proposed by Landau is achieved because the initial thickness of ablative material is so large that can be regarded as infinite. This steady ablation phenomenon may sustain for a certain time until the residual thickness is reduced to a smaller one and the effect of substrate is no longer negligible. Hence, as the ablation proceeds,

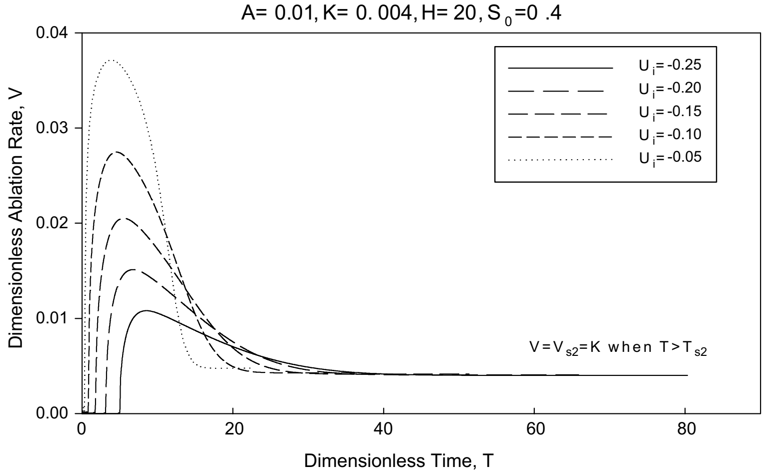


Fig. 6. Effects of initial temperature on the history of ablation rate, resulting from numerical experiments.

the ablation rate would gradually decrease from its maximum, and finally, before the end-of-ablation, approach another steady value $V_{s2}$ which is the new discovered phenomenon as predicted by Eq. (28).

Presented in Figs. 7–11 are detailed results of a numerical experiment corresponding to a typical case of $A = 0.01$, $K = 0.004$, $H = 20$, $U_i = -0.1$, and $S_0 = 0.4$. The associated theoretical solutions are also shown in these figures for comparisons.

Fig. 7 shows the results of function $f(U_1)$ which is related to the temperature distribution in ablative material. The function $f(U_1)$ in this figure is obtained from Eq. (18a) with the following definition:

$$f(U_1) = \ln\left(1 - \frac{U_1 K}{A(HK - 1)}\right) \Big/ (-KS/A) \qquad (45)$$

Thus, the theoretical steady solution of $f(U_1)$ is derived to be $Y/S$ which is the solid line shown in Fig. 7. The various symbols in this figure are corresponding to the "experimental" results at different times. It is shown that, as time goes on, the "experimental" temperature distribution gradually approaches its theoretical steady solution. This proves the correctness of Eq. (18a) for describing the steady temperature distribution in the ablative.

Fig. 8 shows the temperature distribution of substrate as ablation proceeds. The solid line corresponding to the theoretical solution of steady ablation is also plotted for comparison. The various symbols in this figure are corresponding to the "experimental" results at different times. Observe that, as time goes large, the

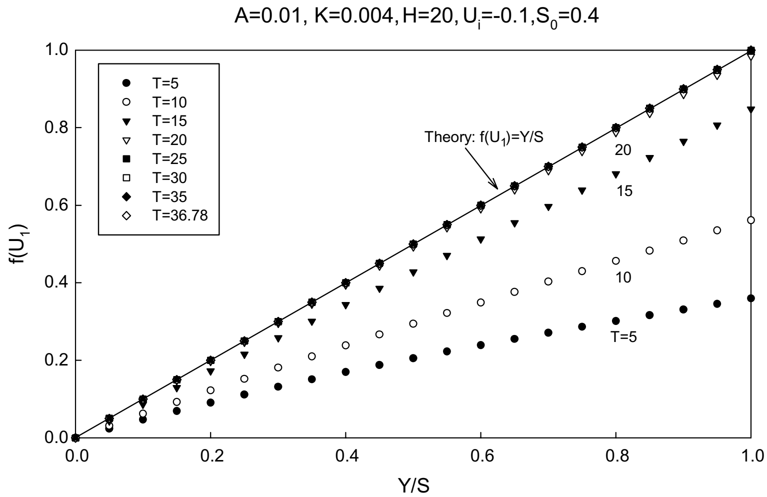


Fig. 7. Comparisons of the theoretical temperature distribution in ablative material with those from a numerical experiment.

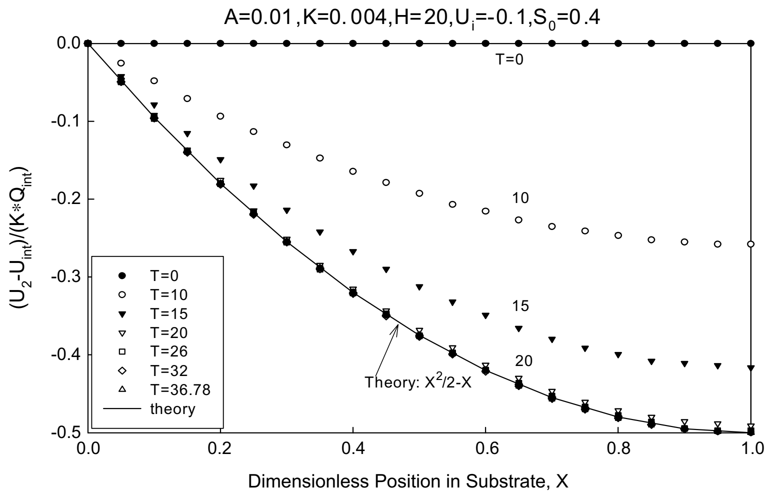


Fig. 8. Comparisons of the theoretical temperature distribution in substrate with those from a numerical experiment.

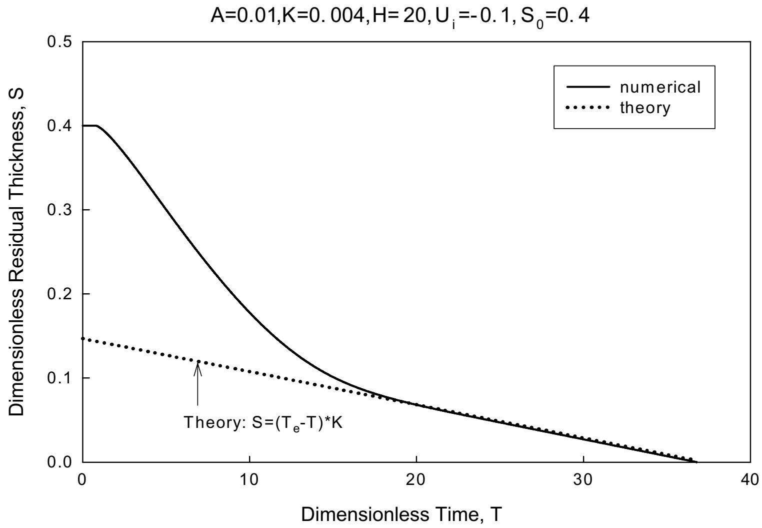


Fig. 9. Comparisons of theoretical to numerical history of residual thickness of ablative material.

temperature distribution in the substrate gradually approaches its theoretical solution, namely, a parabolic function of $X$. This supports the validity of Eq. (31) for describing the temperature distribution in the substrate for steady ablation.

The specific value of 36.78 in Figs. 7 and 8 is actually the time of the last time step just before the end-of-ablation in the numerical experiment for the typical tested case. In fact, the end-of-ablation time could also be predicted theoretically, by Eq. (38), to be 36.69, which is very close to the specific value of 36.78.

Fig. 9 shows a comparison between "experimental" dimensionless residual thickness and its theoretical solution. Note that, as time goes long enough, the experimental result fully matches the theoretical solution and hence Eq. (28) is proved to be correct.

A comparison of "experimental" interfacial temperature $U_{\rm int}$ with its theoretical solution for steady ablation is shown in Fig. 10. It is shown that the "experimental" result perfectly agrees with the theoretical solution as time goes long enough. Therefore, the validity of Eq. (19a) is proved.

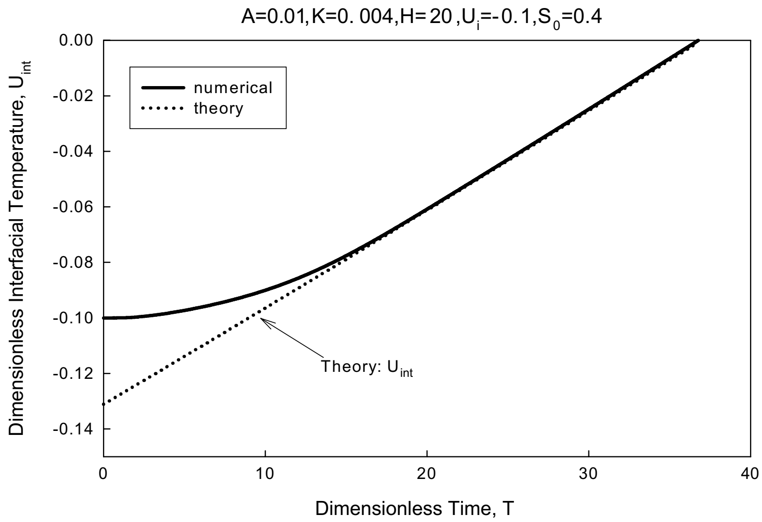


Fig. 10. Comparisons of theoretical to numerical history of interfacial temperature.

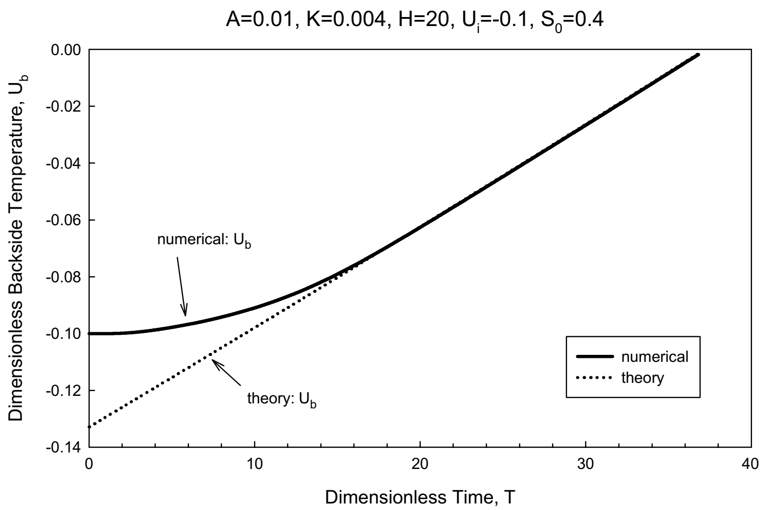


Fig. 11. Comparisons of theoretical to numerical history of backside temperature of substrate.

As shown in Fig. 11, the backside temperature $U_b$ from a numerical experiment is compared with its theoretical solution. Good agreement is obtained as time goes long enough. Hence, the solution of backside temperature, Eq. (32), is considered to be valid.

From Fig. 7–11, it is observed that the end-of-ablation time of this numerical experiment is $T_e = 36.78$, which is nearly identical to its theoretical value $T_e = 36.69$ from Eq. (38). This supports the correctness of Eq. (38).

Fig. 12 shows the relationship between the end-of-ablation time $T_e$ and the two parameters, $H$ and $S_0$. Fig. 13 shows the relationship between $T_e$ and the other two parameters, $K$ and $A$. "Experimental" and theoretical results are both plotted for comparisons in these two figures. The various symbols in these two figures are corresponding to the "experimental" results and the lines corresponding to theoretical solutions (Eq. (38)). As can be seen from these two figures, there are perfectly good agreements between the theoretical and "experimental" results. Hence, Eq. (38) is believed to be valid.

Fig. 14 shows the influences of $A/K^2$ and $S_0$ on $V_e$ which is the ablation rate at the end-of-ablation. "Experimental" and theoretical results are both plotted for comparisons in this figure. For conditions that $A/K^2 \gg 1$, the "experimental" results are observed to

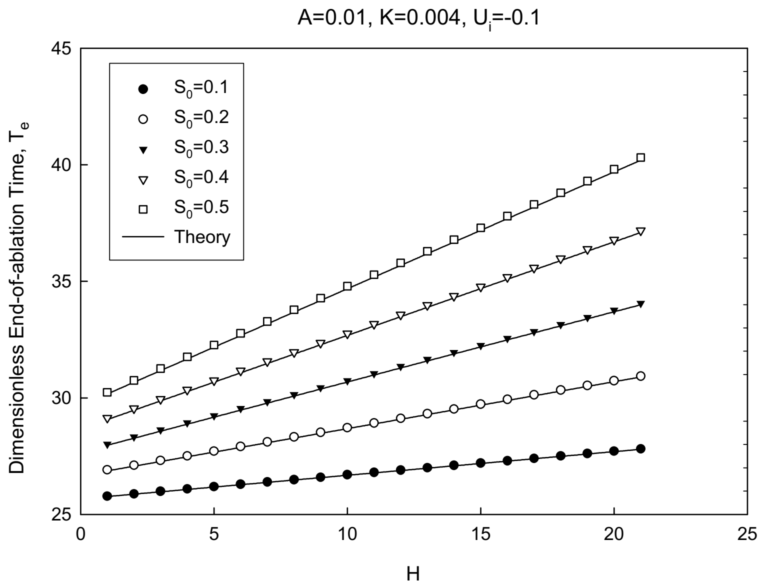


Fig. 12. Effects of $H$ and $S_0$ on end-of-ablation time and comparisons between theoretical and numerical results.

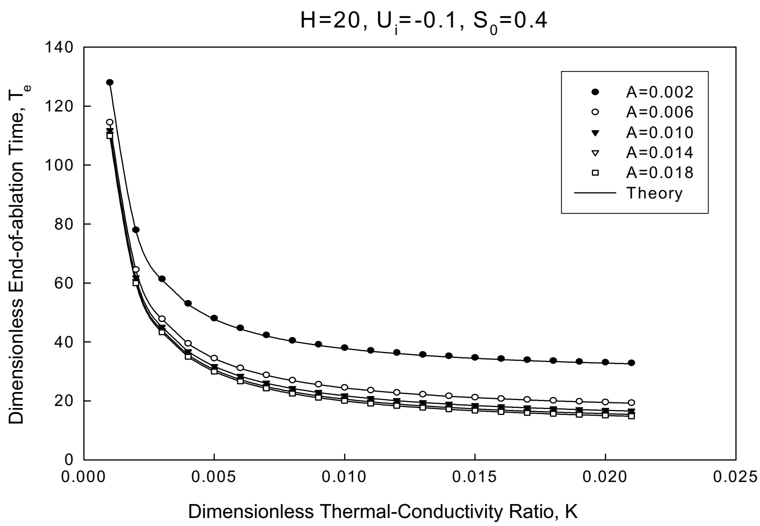


Fig. 13. Effects of $K$ and $A$ on end-of-ablation time and comparisons between theoretical and numerical results.

be in good agreement with the theory for steady ablation. However, while $A/K^2$ become smaller, $V_e$ is observed to gradually deviate from the theoretical value. This means that the theoretical solution for steady ablation is no longer valid and proves the correctness of Eq. (42) as one of the necessary criteria for the occurrence of steady ablation. Also observed from this figure is that the dimensionless initial thickness $S_0$ of ablative material seems to have no influence on $V_e$ no matter whether the steady ablation is achieved or not.

Fig. 15 shows the influence of $A/K^2$ and $A$ on $V_e/V_{s2}$ which means the ratio of ablation rate at the end-of-ablation to theoretical steady ablation rate. It is shown that, while $A/K^2 \gg 1$, the "experimental" values well agree with the theoretical ones (i.e., $V_e/V_{s2} \approx 1$). On the other hand, while $A/K^2$ becomes smaller, the errors would gradually become significant especially for those with larger $A$. This means that Eq. (42) is indeed one of the necessary criteria for the occurrence of steady ablation.

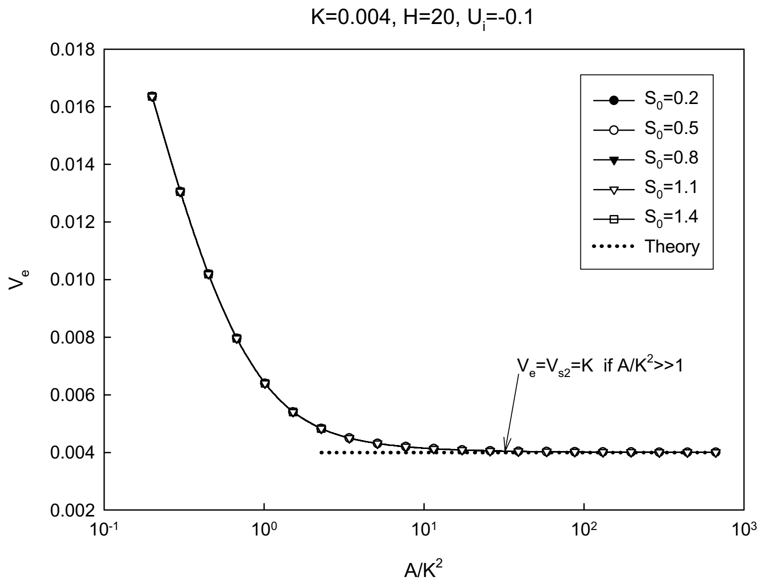


Fig. 14. Effects of $S_0$ and $A/K^2$ on the ablation rate at the end-of-ablation.

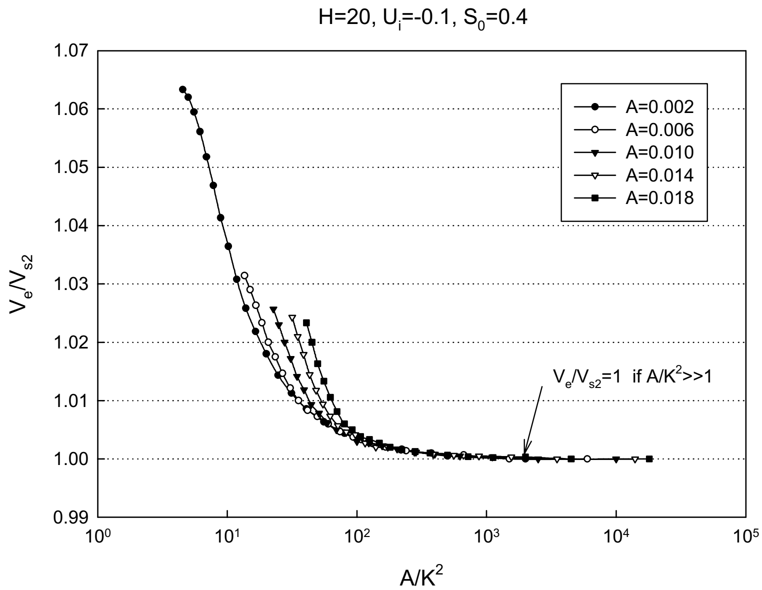


Fig. 15. Effects of $A$ and $A/K^2$ on $V_e/V_{s2}$.

## 5. Conclusions

Based on the results of the current study, the following conclusions can be drawn:

(1) Discovered is a quasi-steady ablation phenomenon on the surface of a two-layer composite which is formed by a layer of ablative material and another layer of non-ablative substrate.
(2) The dimensionless steady ablation rate $V_{s2}$ is proved to be exactly equal to the thermal-conductivity ratio, $K$. Other parameters, such as $A$, $H$, $U_i$, and $S_0$ are shown to have no influence on $V_{s2}$.
(3) While the steady ablation of the current study is achieved, the dimensionless temperature profile in the ablative is shown to be in an exponential form, and is independent of time also, whereas in the substrate the dimensionless temperature

profile is a second-degree polynomial. Moreover, the temperature increasing rate in substrate is proved to be independent of position.

(4) An exact formula of the end-of-ablation time is derived for steady ablation. This dimensionless end-of-ablation time is proved to vary linearly with $H$, $U_i$, $S_0$ and $1/A$.
(5) A criterion ($A/K^2 \gg 1$) for the occurrence of quasi-steady ablation is obtained.
(6) Numerical experiments are accomplished by using a one-dimensional transient numerical model and the "experimental" results are shown to well agree with the theoretical solutions. This proves the correctness of the above theoretical results.
(7) A new method is proposed for measuring the ablation or sublimation heat by means of using Eq. (29) or (34) and measuring the backside temperature response if other parameters are known in advance.
(8) It seems that the necessary criteria for the current steady ablation to occur are very complicated. Except the very one ($A/K^2 \gg 1$) obtained in the current study, there should be some other necessary criteria which must be all satisfied prior to the occurrence of steady ablation. Further studies are needed on this subject.

## Acknowledgement

The author benefited by the many fruitful discussions with Dr. Yao-Hsin Hwang. His contribution to this work is gratefully acknowledged.

## Appendix A. The derivation of substrate's temperature

According to Duhamel's Theorem ([7, p. 159]) and for any time $T \geqslant T_{s2}$, the substrate's temperature $U_2$ can be described by Eq. (24):

$$\frac{(U_2 - U_i)}{K} = \int_{\tau=0}^{T_{s2}} Q_{int}(\tau)\frac{\partial \bar{U}(T-\tau,X)}{\partial T}d\tau + \int_{\tau=T_{s2}}^{T} Q_{int}(\tau)\frac{\partial \bar{U}(T-\tau,X)}{\partial T}d\tau \tag{24}$$

Let the first part ($\tau = 0 \sim T_{s2}$) on the right-hand side be represented by $E_1$ and the second part ($\tau = T_{s2} \sim T$) by $E_2$, i.e.,

$$E_1 \equiv \int_{\tau=0}^{T_{s2}} Q_{int}(\tau)\frac{\partial \bar{U}(T-\tau,X)}{\partial T}d\tau \tag{A.1}$$

$$E_2 \equiv \int_{\tau=T_{s2}}^{T} Q_{int}(\tau)\frac{\partial \bar{U}(T-\tau,X)}{\partial T}d\tau \tag{A.2}$$

The first part ($\tau = 0 \sim T_{s2}$) cannot be obtained in closed form since the theoretical solution of $Q_{int}$ in this transient period is not available whereas the second part ($\tau = T_{s2} \sim T$) can be integrated by using the known solution of $Q_{int}$ in Eq. (21).

In virtue of Eq. (22), we have

$$\frac{\partial}{\partial T}\bar{U}(T-\tau,X) = 1 + 2\sum_{n=1}^{\infty}\exp[-n^2\pi^2(T-\tau)]\cos(n\pi X) \tag{A.3}$$

Substituting the above equation into Eq. (A.1) and assuming $T > T_{s2} + 1$, therefore the exponential terms in the above equation are negligible, thus we have

$$E_1 \cong \int_{\tau=0}^{T_{s2}} Q_{int}(\tau)\,d\tau \tag{A.4}$$

Similarly, by substituting Eq. (A.3) into Eq. (A.2), the following expression is obtained:

$$E_2 = \int_{\tau=T_{s2}}^{T} Q_{int}(\tau)\,d\tau + 2\sum_{n=1}^{\infty}\cos(n\pi X) \times \int_{\tau=T_{s2}}^{T} Q_{int}(\tau)\exp[-n^2\pi^2(T-\tau)]\,d\tau \tag{A.5}$$

Let the first term on the right-hand side be represented by $E_3$ and the second term by $E_4$. By introducing Eq. (21) into the above expression, $E_3$ can be obtained as the following:

$$E_3 = \frac{A}{V_{s2}^2}Q_{int}(T)\left[1 - \exp\left(-\frac{(T-T_{s2})V_{s2}^2}{A}\right)\right] \tag{A.6}$$

whereas $E_4$ is obtained as

$$E_4 = 2(1 - HV_{s2})\sum_{n=1}^{\infty}[\cos(n\pi X)\cdot I_{5n}] \tag{A.7}$$

where $I_{5n}$ is defined as

$$I_{5n} \equiv \int_{\tau=T_{s2}}^{T}\exp\left[-n^2\pi^2(T-\tau) - \frac{S(\tau)V_{s2}}{A}\right]d\tau \tag{A.8}$$

Since it has been known, for steady ablation, that

$$\tau = T_e - \frac{S(\tau)}{V_{s2}} \tag{A.9}$$

Thus we have

$$d\tau = -\frac{dS(\tau)}{V_{s2}} \tag{A.10}$$

Substituting Eqs. (A.9) and (A.10) into Eq. (A.8), we have

$$I_{5n} = \frac{\exp(-SV_{s2}/A)}{[n^2\pi^2 + (V_{s2}^2/A)]}\left\{1 - \exp\left[(S - S_c)\left(\frac{n^2\pi^2}{V_{s2}} + \frac{V_{s2}}{A}\right)\right]\right\} \tag{A.11}$$

where $S_c$ is called "the critical thickness for steady ablation" which is the dimensionless residual thickness at time $T_{s2}$. By assuming $A/V_{s2}^2 \gg 1$, the above equation can be reduced to

$$I_{5n} = \frac{\exp(-SV_{s2}/A)}{n^2\pi^2}\left\{1 - \exp\left[(S - S_c)\frac{n^2\pi^2}{V_{s2}}\right]\right\} \quad (A.12)$$

Substituting the above equation into Eq. (A.7) and combining the known relation $(S_c - S) = (T - T_{s2})V_{s2}$ for steady ablation, we have

$$E_4 = 2Q_{int}\sum_{n=1}^{\infty}\left[\frac{\cos(n\pi X)}{n^2\pi^2}\left\{1 - \exp\left[-(T - T_{s2})n^2\pi^2\right]\right\}\right] \quad (A.13)$$

For brevity of the following derivation, it is assumed that $T > T_{s2} + 1$, therefore, the exponential terms in the above can be neglected. Thus, the above expression can be reduced to

$$E_4 \cong 2Q_{int}\sum_{n=1}^{\infty}\left[\frac{\cos(n\pi X)}{n^2\pi^2}\right] = Q_{int}\left(\frac{X^2}{2} - X + \frac{1}{3}\right) \quad (A.14)$$

According to the definitions in the above derivation, we have

$$\frac{(U_2 - U_i)}{K} = E_1 + E_2 = E_1 + E_3 + E_4 \quad (A.15)$$

Hence, by introducing Eqs. (A.4), (A.6) and (A.14) into the above equation, it is obtained:

$$\frac{(U_2(T,X) - U_i)}{KQ_{int}(T)} \cong \frac{1}{Q_{int}(T)}\int_{\tau=0}^{T_{s2}} Q_{int}(\tau)\,d\tau + \frac{A}{V_{s2}^2}\left[1 - \exp\left(-\frac{(T - T_{s2})V_{s2}^2}{A}\right)\right] + \frac{X^2}{2} - X + \frac{1}{3} \quad (25)$$

This is Eq. (25) which describes the temperature of substrate.

## References


[1] H.G. Landau, Heat conduction in a melting solid, Quart. Appl. Math. 8 (1) (1950) 81–94.

[2] T.R. Goodman, The heat-balance integral and its application to problems involving a change of phase, Trans. ASME 80 (1958) 335–342.

[3] T.R. Goodman, Application of integral methods to transient nonlinear heat transfer, in: T.F. Irvine Jr., J.P. Hartnett (Eds.), Advances in Heat Transfer, vol. 1, Academic Press, New York, 1964, pp. 51–122.

[4] T.F. Zien, Approximate calculation of transient heat conduction, AIAA J. 14 (1976) 404–406.

[5] T.F. Zien, Integral solutions of ablation problems with time-dependent heat flux, AIAA J. 16 (12) (1978) 1287–1295.

[6] R.L. Potts, Application of integral methods to ablation charring erosion, a review, J. Spacecraft Rockets 32 (2) (1995) 200–209.

[7] G.E. Myers, Analytical Methods in Conduction Heat Transfer, McGraw-Hill, 1971.